Axial Mixing in Pipes

L. J. TICHACEK, C. H. BARKELEW, and THOMAS BARON

Shell Development Company, Emeryville, California

The problem of axial mixing in straight pipes is analyzed by a modification of G. I. Taylor's analysis. The treatment presented here includes the effect of both Schmidt and Reynolds numbers throughout the turbulent-flow range. All applicable data on flow of gases and liquids are found to confirm the validity of the method.

The analysis indicates that axial mixing increases rapidly as the flow approaches the laminar region, especially for liquids, and that pipe roughness causes a relatively small increase in axial mixing. Turbulent eddy diffusion in the axial direction has a negligible effect.

The results of the analysis are applicable to those systems wherein the kinematic viscosity of the flowing mixture does not vary greatly from one region to another and in which the concentration region of interest is spread out along a sufficient length of pipe. These limitations are broad enough to permit most practical problems to be treated by the method.

The passage of fluid through a pipe is accompanied by mixing in the axial direction. This effect, which can be observed, for instance, by noting the dispersion of tracers, results in intermixing of products in pipe lines, in decreasing the driving force in tubular reactors, and in diminishing the sharpness of signals in tracer experiments.

In unbroken straight pipes axial mixing is due to diffusion in the axial direction because of molecular or turbulent motions and to the relative axial motion of fluid elements at different radial positions. However, the effects of axial molecular and turbulent diffusion are negligible compared with the interpenetration due to relative motion. The latter depends greatly on the shape of the velocity profile and the rate of radial diffusion. The more nearly the velocity profile approaches that for plug flow, the smaller is the amount of axial mixing. A high rate of radial diffusion tends to keep the concentration radially uniform; the different radial fluid elements then have more nearly the same composition and in moving with respect to one another, cause less severe mixing. Thus axial mixing is pronounced in the case of laminar flow, where the flow is least pluglike and where radial diffusion is small (molecular instead of turbulent).

The first analysis of axial mixing based on radial variation of the velocity was made by G. I. Taylor (*17* to *19*), who treated first the case of laminar flow in capillary tubes and later the case of turbulent flow in pipes. His treatment of turbulent flow is valid only for high Reynolds numbers because he used a velocity profile valid only when the laminar sublayer and transition layers are negligibly small. Experimental results support this treatment for Reynolds numbers greater than 20,000.

In the calculations presented here, Taylor's method is refined and extended to cover the whole range of turbulent flows. The chief differences introduced here are the inclusion of the effect of molecular diffusion and the use of experimental velocity profiles rather than a generalized profile. It was necessary to rearrange the mathematical expressions so as to increase the accuracy of calculations based on measured profiles. Also an error analysis is presented which shows that the calculations are valid for most practical purposes.

Mixing because of bends, elbows, valves pumps, etc., must be considered as an effect additional to that discussed here.

THEORETICAL ANALYSIS

In Taylor's analysis the differential equation for transport of material from a radial element of a disk moving with average stream velocity V was written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha r\frac{\partial c}{\partial r}\right) = (u - V)\frac{\partial c}{\partial x} + \frac{\partial c}{\partial t}, \quad (1)$$

where

r = distance from center of pipe

α = diffusivity in radial direction

u = axial velocity at r

x = axial distance from reference plane of velocity V

This equation applies to a point moving along with the mean velocity V and neglects molecular and eddy diffusion in the axial direction. To obtain a first approximation to the solution of the mixing problem it is assumed that $\partial c/\partial x$ is constant in the pipe. It follows that $\partial c/\partial t = 0$, and

$$\alpha r\frac{\partial c}{\partial r} = \frac{\partial c}{\partial x}\int_0^r (u - V)r'\,dr' \quad (2)$$

Further integration gives c_r, the concentration at position r:

$$c_r - c_a = \frac{\partial c}{\partial x}\int_a^r \frac{1}{\alpha r'}\,dr' \cdot \int_0^{r'} (u - V)r''\,dr'' \quad (3)$$

The net transport past any reference plane moving with the fluid with the mean velocity V is

$$Q = 2\pi\int_0^a (c_r - c_a)(u - V)r\,dr \quad (4)$$

where a is the pipe radius. Inspection of Equations (3) and (4) shows that the net transport is proportional to the axial concentration gradient; therefore an effective coefficient of axial diffusion E can be defined in the usual way:

$$Q = -\pi a^2 E\frac{\partial c_m}{\partial x} \quad (5)$$

Thus one obtains from Equations (3), (4) and (5),

$$E = -\frac{2}{a^2}\int_0^a (u - V)r\,dr\int_a^r \frac{1}{\alpha r'}\,dr' \cdot \int_0^{r'} (u - V)r''\,dr'' \quad (6)$$

Taylor used the equivalent of this expression to find the apparent axial diffusivity E, evaluating u and α as functions of r

from a universal velocity profile. The analysis may be extended by using measured velocity profiles, and accounting for molecular transport in α.

$$\alpha = \epsilon_D + \mathbf{D} \qquad (7)$$

A similar expression can be applied for the transport of momentum:

$$\tau = -(\epsilon_m + \nu)\rho\frac{\partial u}{\partial r} \qquad (8)$$

If it is assumed that $\epsilon_D = \epsilon_m$, then from Equations (7) and (8)

$$\alpha = -\frac{\tau}{\rho}\frac{\partial r}{\partial u} - \nu\left(1 - \frac{1}{N_{Sc}}\right) \qquad (9)$$

where N_{Sc} is the Schmidt number. τ can be expressed in terms of τ_0, the shear stress at the wall, or λ, the Moody friction factor ($\lambda/4 = f$, the Fanning friction factor), as follows:

$$\tau = \frac{r}{a}\tau_0 = \frac{r}{a}\cdot\frac{1}{8}\lambda\rho V^2 \qquad (10)$$

Equations (6), (9), and (10) then reduce to

$$E = \frac{2}{a^2}\int_0^a \frac{dr}{-\frac{\lambda}{8}\frac{r^2}{a}V^2\frac{\partial r}{\partial u} - r\nu\left(1 - \frac{1}{N_{Sc}}\right)} \cdot \int_0^r (u - V)r'\,dr' \int_0^r (u - V)r''\,dr'' \qquad (11)$$

where order of integration has also been changed. The substitution $z = (r/a)^2$ yields the final expression

$$\frac{E}{DV} = \frac{2}{\lambda}\int_0^1 \left\{\frac{1}{-\frac{\partial z}{\partial(u/V)} - \frac{32}{\lambda N_{Re}}\left[1 - \frac{1}{N_{Sc}}\right]}\right\}\frac{dz}{z}\left\{\int_0^z\left(\frac{u}{V} - 1\right)dz'\right\}^2 \qquad (12)$$

The variation of u/V with z (i.e., the velocity profile) is dependent only on the Reynolds number and pipe roughness. Thus E/DV depends on the Reynolds number, friction factor (or the roughness), and Schmidt number. It should be understood that this analysis cannot be applied to the mixing of two fluids flowing one behind the other unless the kinematic viscosity is approximately the same in each fluid; i.e., the fluids should not have Reynolds numbers so dissimilar as to cause substantial differences between their velocity profiles.

Evaluation can be done numerically as follows. For a chosen Reynolds number the velocity profile is plotted in the form u vs. z (i.e., vs. r^2/a^2). The values of $\partial u/\partial z$ and $A(z)$ are then calculated for various values of z:

$$A(z) = \int_0^z u\,dz'$$

which also yields $V = A(1)$. After tabulation of values of $[(1/V)A(z) - z]$ and $[(1/V)A(z) - z]^2$, the final integrand

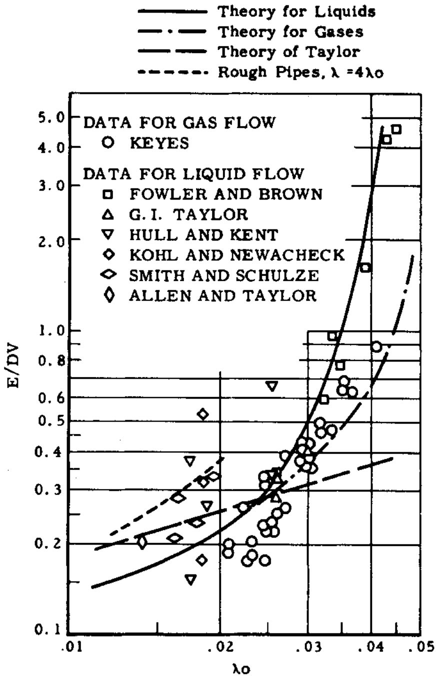

Fig. 1. Axial mixing coefficient vs. smooth-tube friction factor.

$$\frac{\frac{1}{z}\left[\frac{1}{V}A(z) - z\right]^2}{-V\frac{\partial z}{\partial u} - \frac{2}{\lambda N_{Re}}\left[1 - \frac{1}{N_{Sc}}\right]}$$

is found to be a reasonably smooth function of z and thus can be integrated according to Simpson's method. These calculations were made for flow in smooth pipes over a range of Reynolds numbers by use of the velocity profile data of several investigators (*2*, *7* to *11*, *13*, *15*).

Equation (12) shows that as long as the Schmidt number is larger than 100 it does not appreciably affect the value of E/DV. For low Schmidt numbers around $N_{Sc} = 1$ the value of E/DV is more sensitive to variations in N_{Sc}. Calculations indicate, however, that differences in N_{Sc} normally encountered in gases will cause variations in E/DV no greater than those due to experimental error in determining E/DV. Accordingly, two cases were considered: one in which the Schmidt number was taken as 1.0, which is typical of gases, the other in which the Schmidt number was taken larger than 100, which is typical of liquids. For both of these cases the values of E/DV calculated from Equation (12) are shown in Figure 1 as a smooth curve with λ_0 as abscissa where λ_0 is the (Moody) friction factor in a smooth tube.

DATA ON AXIAL MIXING

The results of observations of axial mixing in liquids as reported by a number of investigators (*1*, *3*, *4*, *6*, *14*, *18*) are plotted on Figure 1. The data scatter somewhat, but there is substantial agreement with the theory. The data of several observers (*12*, *14*) were discarded completely or in part because of the inapplicability of this theory under the conditions of certain experiments, i.e., one liquid much more viscous than the other, turbulence not fully developed, or coils in the pipe which cause secondary flow and distortion of the velocity profile.

Observations of axial mixing in gases have been reported by Keyes (*5*), who attempted to measure the thickness of the hypothetical "film" in terms of which mass transfer resistance is usually expressed. Unfortunately the thickness of a "holdup film" calculated from his data demonstrably bears no unique relation to the thickness of the "mass transfer film." Furthermore, the agreement of his data with theory as shown in Figure 1 indicates that there is no need to introduce

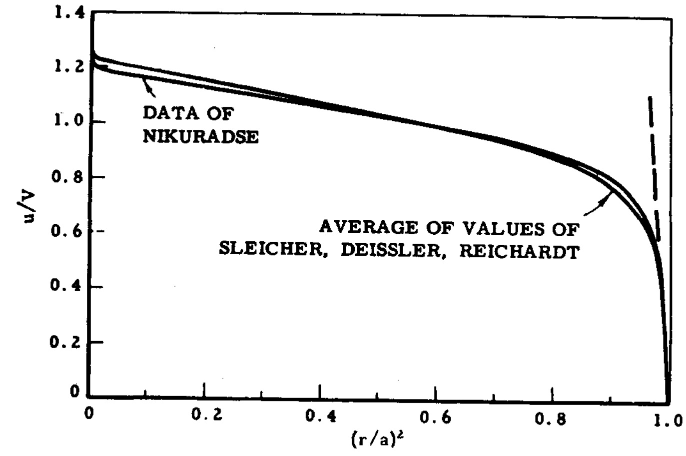

Fig. 2. Comparison of several sets of velocity profile data at N_{Re} — 42,000.

hypothetical films to describe axial dispersion.

EFFECT OF PIPE ROUGHNESS

An attempt was made to evaluate the effect of pipe roughness on axial mixing. Available velocity profile data (*9*) are not accurate enough, especially near and in the layer of roughening, to show more than a qualitative effect of roughness on axial dispersion.

Pipe roughness produces two effects which partially compensate each other: (1) the velocity profile is less pluglike, and (2) turbulence near the walls, and thus radial diffusion, is greater. Usually the first effect dominates, with the result that roughness tends to increase the axial mixing. An approximate curve is partially drawn in Figure 1 for the case of pipe roughnesses such that $\lambda = 4\lambda_0$. (λ is the actual friction factor; λ_0 is the friction factor in a smooth tube for flow at the same Reynolds number.)

TABLE 1 VELOCITY PROFILES USED IN CALCULATIONS

$N_{Re} \times 10^{-3}$	2.56	3.04	4.0	6.0	10.0	40.0	43.4	91.	205.	500.	2790.
u/V at $(r/a)^2$											
0	1.463	1.367	1.320	1.308	1.276	1.241	1.213	1.213	1.190	1.166	1.140
.1	1.411	1.320	1.268	1.244	1.222	1.195	1.167	1.165	1.146	1.135	1.113
.2	1.347	1.280	1.235	1.201	1.182	1.156	1.134	1.130	1.116	1.109	1.093
.3	1.283	1.239	1.197	1.164	1.147	1.117	1.105	1.094	1.088	1.081	1.072
.4	1.215	1.193	1.156	1.128	1.109	1.078	1.070	1.058	1.060	1.053	1.050
.5	1.138	1.140	1.111	1.086	1.069	1.039	1.033	1.023	1.031	1.025	1.026
.6	1.028	1.062	1.054	1.033	1.024	.996	.994	.983	.997	.994	1.000
.7	.866	.936	.976	.968	.967	.946	.952	.939	.954	.959	.970
.8	.633	.731	.841	.860	.880	.880	.897	.888	.899	.913	.922
.85	.486	.581	.710	.762	.804	.836	.861	.853	.862	.883	.891
.9	.324	.390	.485	.611	.677	.776	.808	.807	.814	.839	.847
.95	.162	.195	.248	.348	.441	.670	.709	.738	.740	.765	.784
1.0	0	0	0	0	0	0	0	0	0	0	0
$(E/DV)_{gases}$	1.64	.98	.64	.49	.40	.282	.185	.188	.175	.153	.135
$(E/DV)_{liquids}$			2.82		.53		.185	.188	.175	.153	.135
λ_0	.048	.044	.040	.035	.0305	.0215	.021	.018	.0155	.013	.010

LIMITATIONS OF THE THEORY AND OF ITS APPLICATION

Accuracy of Evaluation

It should be pointed out that the accuracy with which E/DV can be evaluated depends sensitively on the accuracy of the velocity profile. This is illustrated by Figure 2, in which two slightly different sets of velocity profile data are given for Reynolds numbers of about 42,000. Although the magnitudes of the velocities u/V at various radial positions are seldom in disagreement by more than 3% between one set of data and the other, E/DV nevertheless depends on more than the second power of the difference $(u/V - 1)$, and as a consequence there is a 50% disagreement between values of E/DV obtained from the two profiles. Since the value of E/DV does depend very markedly on the profile used, the values of u/V employed for calculations in this paper are presented in Table 1. These values are the result of smoothing and averaging velocity profiles taken by many investigators (*2*, *7* to *11*, *13*, *15*).

It should also be pointed out that the calculation for liquids at Reynolds numbers below 10,000 is very difficult. In this region the nearly laminar film at the pipe walls becomes increasingly important and radial diffusion there is difficult to evaluate. Even though velocity profiles may indicate a laminar region, nevertheless the flow may be intermittently turbulent and the effective radial diffusivity may be much larger than for purely molecular diffusion. [Such an effect is clearly exposed by the work of Stirba and Hurt (*16*).] It is in this region that axial mixing data are most important. Fortunately, and perhaps fortuitously, the data available for this region ($\lambda_0 > 0.03$) confirm the calculations.

Assumptions in the Theory

Equation (2) is based on the assumption that $\partial c/\partial x$ is constant throughout the pipe. For some systems, therefore, this first-order approximation limits the accuracy with which E/DV can be calculated accordingly to Equation (12). Greater accuracy in evaluating E/DV can be attained by making a first-order approximation to the variation of $\partial c/\partial x$ in the pipe. This refinement is here used to indicate the validity of Equation (12) under various conditions.

The differential equation for variation of the axial concentration gradient is obtained by differentiating Equation (1) with respect to x:

$$\frac{\partial^2 c}{\partial t \partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(\alpha r \frac{\partial^2 c}{\partial r \partial x}\right) - (u - V)\frac{\partial^2 c}{\partial x^2} \qquad (13)$$

The difference between the value of the axial gradient at any radial position and the value at the center of this pipe is then found by integrating Equation (13):

$$\left.\frac{\partial c}{\partial x}\right|_r - \left.\frac{\partial c}{\partial x}\right|_0 = \int_0^r \frac{1}{\alpha r'}\,dr' \cdot \int_0^{r'} (u - V)\frac{\partial^2 c}{\partial x^2} r''\,dr'' + \int_0^r \frac{1}{\alpha r'}\,dr' \int_0^{r'} \frac{\partial^2 c}{\partial t \partial x} r''\,dr'' \qquad (14)$$

Since a first-order approximation of $\partial c/\partial x\,|_r$ is desired, the quantities $\partial^2 c/\partial x^2$ and $\partial^2 c/\partial t\,\partial x$ are replaced by their mean values, $\partial^2 c_m/\partial x^2$ and $\partial^2 c_m/\partial t \partial x$, respectively. Substitution for the latter term is made according to the relation

$$\frac{\partial}{\partial x}\frac{\partial c_m}{\partial t} = \frac{\partial}{\partial x}\left(E\,\frac{\partial^2 c_m}{\partial x^2}\right) \qquad (15)$$

With these substitutions Equation (14) is reduced to

$$\left.\frac{\partial c}{\partial x}\right|_r - \left.\frac{\partial c}{\partial x}\right|_0 = \frac{\partial^2 c_m}{\partial x^2}\int_0^r \frac{1}{\alpha r'}\,dr' \int_0^{r'} (u - V) r''\,dr'' + E\,\frac{\partial^3 c_m}{\partial x^3}\int_0^r \frac{r'}{2\alpha}\,dr' \qquad (16)$$

For convenience the term $\partial c/\partial x\,|_0$ is related to $\partial c_m/\partial x$ by the calculation of

$$\frac{\partial c_m}{\partial x} - \left.\frac{\partial c}{\partial x}\right|_0 = \frac{1}{a^2}\int_0^a \left\{\left.\frac{\partial c}{\partial x}\right|_r - \left.\frac{\partial c}{\partial x}\right|_0\right\} 2r\,dr \qquad (17)$$

The equivalent of $\partial c/\partial x$ from Equation (16) is substituted for $\partial c/\partial x$ in Equation (1). The term $\partial c/\partial t$ in Equation (1) is approximated by $\partial c_m/\partial t$, which can be replaced by $E(\partial^2 c_m/\partial x^2)$. Modified in this way Equation (1) is the basis of a development paralleling that in Equations (2) to (12). The result is a second-order approximation to the value of the axial mixing coefficient (designated E'). For Schmidt numbers near 1.0 or Reynolds numbers greater than 10,000 the error in the first-order approximation for E is estimated from

$$\frac{E'}{DV} - \frac{E}{DV} = -\frac{2}{\lambda}\int_0^1 \left(\frac{u}{V} - 1\right) dz$$

$$\cdot \int_1^z - \frac{\partial u/V\, dz'}{\partial z' z'}$$

$$\cdot \int_0^{z'} \left\{\left(\frac{u}{V} - 1\right)\frac{c_m''}{c_m'}\frac{4a}{\lambda}\left[\Theta(z'') - \overline{\Theta}\right]\right.$$

$$- \frac{c_m'''}{c_m'}\frac{8a^2}{\lambda}\frac{E}{DV}\left(\frac{u}{V} - 1\right)^2$$

$$\left. + \frac{E}{DV}\, 2a\, \frac{c_m''}{c_m'}\right\} dz'' \tag{18}$$

where c_m', c_m'', and c_m''' are used for the first, second, and third derivatives, respectively, with regard to x, and where

$$\Theta(z) = \int_0^z \frac{-\partial(uV)}{\partial z'}\frac{dz'}{z'}$$

$$\cdot \int_0^{z'} \left(\frac{u}{V} - 1\right) dz'' \tag{19}$$

and

$$\overline{\Theta} = \int_0^1 \Theta(z)\, dz \tag{20}$$

For a given Reynolds number and friction factor, Equation (18) relates the approximate error in the plotted values of E/DV to the various axial derivatives of c_m, these derivatives being expressed in terms of pipe diameters. This error has been evaluated at two Reynolds numbers, as follows:

$$\left.\begin{array}{l} \text{For } N_{Re} = 10{,}000 \\ \dfrac{E' - E}{E} = 2.8a\dfrac{c_m''}{c_m'} \\ \qquad\qquad - 30a^2\dfrac{c_m'''}{c_m'} \\ \text{For } N_{Re} = 205{,}000 \\ \dfrac{E' - E}{E} = 3.7a\dfrac{c_m''}{c_m'} \\ \qquad\qquad - 16a^2\dfrac{c_m'''}{c_m'} \end{array}\right\} \tag{21}$$

It is seen that the accuracy of the first-order approximation depends on the shape and scale of the concentration wave along the pipe axis as characterized by the ratios a^2c_m''/ac_m' and a^3c_m'''/ac_m'. The term c_m''/c_m' in Equations (21) indicates that E' is greater on the back (upstream) side of a concentration pulse than on the front side; the experiments of Taylor (*18*) reveal such a "tailing off" behind a short concentration pulse.

The term c_m'''/c_m' does not change sign from the front to the back of the concentration pulse or contaminated region; i.e., c_m'''/c_m' is roughly symmetrical in x. This term therefore describes an over-all discrepancy between predicted values of E/DV and those obtained by experiments on gross spreading of material in pipes. Evaluation of this effect for the case of flow at a Reynolds number of 10,000 was made for three types of axial concentration profiles: error function (one liquid following another), sinusoidal (Keyes experiments), and error-function derivative (Taylor's pulse-injection experiments). It was found that Equation (12) could be applied with less than 25% error (less than the usual scatter of data in such work) to those cases of mixing in which the characteristic concentration length was greater than 50 to 100 pipe diameters. For smaller characteristic lengths, E/DV can be substantially larger than predicted by Equation (12).

TABLE 2. EFFECT OF AXIAL EDDY DIFFUSIVITY

$N_{Re} \times 10^{-3}$	ϵ_{max}/DV	E/DV	ϵ_{max}/E
2.56	0.0079	1.64	0.005
4.0	0.0135	0.65	0.021
10.0	0.0105	0.38	0.028
205	0.0066	0.18	0.037
2800	0.0055	0.14	0.039

(The characteristic length is taken as the wave length for the case with sinusoidal variation of concentration, or as the "contaminated length" for the other two cases. *Contaminated length* is here defined as the length of the region in which the concentration is between 1 and 99% of the maximum concentration of either fluid.) It can therefore be concluded that the system studied by Keyes and likewise any practical pipe-line mixing problem will have axial concentration gradients of a size and variation that will allow treatment by this theory.

EFFECT OF AXIAL EDDY DIFFUSION

It is assumed in this development that the effects of axial turbulent diffusion are negligible when compared with the mixing caused by radial differences in the velocity. An estimate of the magnitude of axial turbulent diffusion can be obtained from the values of the radial turbulent diffusivity. In Table 2 the coefficients of axial mixing are compared with the values of the eddy diffusivity at the radial position where eddy diffusivity is greatest. In all cases the eddy diffusivity, ϵ, is seen to be negligible.

NOTATION

a = pipe radius
c = concentration
c_m = mean concentration across pipe cross section
$\left.\begin{array}{l} c_m' \\ c_m'' \\ c_m''' \end{array}\right\}$ = first, second, and third derivatives of c_m with respect to x
D = pipe diameter
$\mathbf{D}$ = molecular diffusion coefficient
E = effective axial mixing coefficient
E' = effective axial mixing coefficient calculated without neglect of radial variation of $\partial c/\partial x$
N_{Re} = Reynolds number, DV/ν
N_{Sc} = Schmidt number, $\nu/\mathbf{D}$
Q = net transport of solute past reference plane with velocity V
r = radial coordinate
t = time
u = time-averaged velocity at a point in the pipe
V = mean velocity across pipe
x = axial coordinate, reckoned from a reference plane moving with velocity V
z = transformed radial coordinate, $(r/a)^2$

Greek Letters

α = total effective radial diffusion coefficient, Equation (7)
ϵ_D = coefficient of eddy transport of mass
ϵ_m = coefficient of eddy transport of *momentum*
λ = friction factor (Moody)
λ_0 = friction factor in a smooth tube
ν = kinematic viscosity
ρ = density
τ = shear stress
τ_0 = shear stress at the wall

LITERATURE CITED

1. Allen, C. M., and E. A. Taylor, *Trans. Am. Soc. Mech. Engrs.*, **45**, 285 (1923).
2. Deissler, R. G., *Natl. Advisory Comm. Aeronaut. Technical Note* 2138 (1950).
3. Fowler, F. C., and G. G. Brown, *Trans. Am. Inst. Chem. Engrs.*, **39**, 491 (1943).
4. Hull, D. E., and J. W. Kent, *Ind. Eng. Chem.*, **44**, 2745 (1952).
5. Keyes, J. J., *A.I.Ch.E. Journal*, **1**, 305 (1955).
6. Kohl, J., and R. L. Newacheck, paper presented at the fall 1953 meeting of the Am. Soc. Mech. Engrs. Petroleum Division.
7. Laufer, John, *Natl. Advisory Comm. Aeronaut. Technical Note* 2954 (1953).
8. Nikuradse, J., *Forschungsheft* 356 (1932).
9. *Ibid.*, 361 (1933).
10. Reichardt, H., *Z. Ang. Math. u. Mech.*, **31**, 208 (1951).
11. Senecal, V. E., and R. R. Rothfus, *Chem. Eng. Progr.*, **49**, 533 (1953).
12. Shipley, J. R., *Pipe Line News*, p. 31 (December, 1951).
13. Sleicher, C. A., Ph.D. thesis, Univ. Michigan, Ann Arbor (1955).
14. Smith, S. S., and R. K. Schulze, *Petroleum Eng.*, p. 330 (October, 1948).
15. Stanton, T. E., *Proc. Roy. Soc.* (*London*), **A85**, 366 (1911).
16. Stirba, Clifford, and D. M. Hurt, *A.I.Ch.E. Journal*, **1**, 178 (1955).
17. Taylor, G. I., *Proc. Roy. Soc.* (*London*), **A219**, 186 (1953).
18. *Ibid.*, **A223**, 446 (1954).
19. *Ibid.*, **A225**, 473.